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A Note on the Optimal Recovery of Functions in H^∞

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The error of the best approximation of functions $f \in H^\infty$ on the basis of given Hermitian data $\{f^{(\lambda)}(x_k), k=1, ..., n, \lambda=0, ..., v_k-1\}$ is expressed by the Blaschke product $B(\bar{x}; t)$ with zeros $\bar{x}=(x_1, ..., x_n)$ of multiplicities $v_1, ..., v_n$, respectively. Given $(v_k)_1^n$, we prove the uniqueness of the nodes $\bar{x}^*$ which are optimal of type $(v_1, ..., v_n)$, i.e., which minimize the uniform norm of $B(\bar{x}; \cdot)$ in $[a, b] \subset (-1, 1)$ over $a \leqslant x_1 \leqslant \cdots \leqslant x_n \leqslant b$. The extremal function $B(\bar{x}^*; t)$ is characterized by an oscillation property. Finally, a comparison theorem is proved, showing the dependence of the error on the order of the derivatives used in the information data.

1. INTRODUCTION

As usual H^∞ is the Banach space of bounded analytic functions in the disk $D := \{z: |z| < 1\}$ with norm

$$\|f\|_\infty := \sup\{|f(z)|: z \in D\}.$$

Let $[a, b]$ be a given subinterval of $(-1, 1)$. We shall denote here by $\|f\|$ the uniform norm of f in $[a, b]$.

Our paper is concerned with the problem of best approximation of functions f from H^∞ on the basis of the data $(l_1(f), ..., l_N(f))$, where $\{l_k\}_1^N$ are fixed continuous linear functionals. We first recall some facts and definitions.

Denote by $B(H^\infty)$ the unit ball in H^∞, i.e.,

$$B(H^\infty) := \{f \in H^\infty: \|f\|_\infty \leqslant 1\}.$$

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Suppose that x is fixed in $[a, b]$. Any mapping S of the set $\{(l_1(f), ..., l_N(f)) : f \in H^\infty\}$ into the complex plane $\mathbb{C}$ defines a method of approximation

$$f(x) \approx S(l_1(f), ..., l_N(f))(x) \tag{1}$$

with error

$$R_S(l_1, ..., l_N)(x) := \sup_{f \in B(H^\infty)} |f(x) - S(l_1(f), ..., l_N(f))(x)|. \tag{2}$$

The method S^* for which

$$R_{S^*}(l_1, ..., l_N)(x) = R(l_1, ..., l_N)(x) := \inf_S R_S(l_1, ..., l_N)(x)$$

is said to be a best method of recovery of $f(x)$ on the basis of the information $(l_1(f), ..., l_N(f))$.

It follows from a well-known general result due to Smolyak [9] (see also [1] or [6, 10] for extensions) that

$$R(l_1, ..., l_N)(x) = \sup\{|f(x)| : f \in B(H^\infty), l_1(f) = \cdots = l_N(f) = 0\}. \tag{3}$$

Moreover, the minimal error $R(l_1, ..., l_N)(x)$ is achieved by a linear method of the form

$$f(x) \approx \sum_{k=1}^{N} c_k(x)\, l_k(f). \tag{4}$$

Letting x to run over $[a, b]$, we obtain a linear approximation scheme with a nice extremal property. The quantity

$$R(l_1, ..., l_N) := \|R(l_1, ..., l_N)(\cdot)\| \tag{5}$$

is the error of (4) in H^∞.

It is a natural question to ask for those functionals that minimize $R(l_1, ..., l_N)$. It turns out that the function evaluations $f(x_1^*), ..., f(x_N^*)$ at some special points $a < x_1^* < \cdots < x_N^* < b$ form a system of extremal functionals. We prove this in Section 3.

Next we concentrate on the case when the information $(l_1, ..., l_N)$ is Hermitian data $\{f^{(\lambda)}(x_k),\ k = 1, ..., n,\ \lambda = 0, ..., \nu_k - 1\}$, where $a \leqslant x_1 < \cdots < x_n \leqslant b$ and $(\nu_k)_1^n$ are fixed natural numbers, $\nu_1 + \cdots + \nu_n = N$. The Blaschke products

$$B(\bar{x}; z) := \prod_{i=1}^{n} \left(\frac{z - x_i}{1 - z x_i}\right)^{\nu_i}$$

appear prominently in the study of this important case. It follows easily (see [7]) from the maximum of the modulus theorem that for each $z \in D$,

$$|B(\bar{x}; z)| = \sup\{|f(z)| : f \in B(H^\infty), f^{(\lambda)}(x_k) = 0, k = 1, ..., n, \lambda = 0, ..., \nu_k - 1\}.$$

Thus, in view of the definitions (3) and (5), the best choice of the points of evaluation is that one which minimizes the uniform norm of the Blaschke product $B(\bar{x}; \cdot)$ in $[a, b]$, over the set $a \leqslant x_1 < \cdots < x_n \leqslant b$.

DEFINITION. The nodes $\bar{x}^* = (x_1^*, ..., x_n^*)$ are said to be optimal of type $(\nu_1, ..., \nu_n)$ in H^∞ if $a \leqslant x_1^* < \cdots < x_n^* \leqslant b$ and

$$\|B(\bar{x}^*; \cdot)\| = \inf\{\|B(\bar{x}; \cdot)\| : a \leqslant x_1 < \cdots < x_n \leqslant b\}.$$

We prove in Theorem 2 below the uniqueness of the optimal nodes. As an auxiliary result we show the existence and uniqueness of the Blaschke product $B(\bar{x}; z)$ that has preassigned local extremums.

2. MAIN RESULTS

We start with a simple lemma.

LEMMA 1. *Let $(x_k)_1^n$ be arbitrary points such that $-1 < x_1 < \cdots < x_n < 1$. Then the functions*

$$\varphi_i(t) := \frac{1}{(1 - t x_i)(t - x_i)}, \qquad i = 1, ..., n,$$

form a Tchebycheff system on $A := (-1, 1)/\{x_1, ..., x_n\}$.

Proof. Assume the contrary. Then there exist distinct points $(\tau_j)_1^n$ in A and a non-zero linear combination φ of $\{\varphi_i\}_1^n$ such that $\varphi(\tau_j) = 0$ for $j = 1, ..., n$. Since

$$\varphi_k(1/\tau_j) = \tau_j^2 \varphi_k(\tau_j),$$

we conclude that

$$\varphi(1/\tau_j) = \tau_j^2 \varphi(\tau_j) = 0 \qquad \text{if} \quad \tau_j \neq 0. \tag{6}$$

On the other hand, $\varphi(t) = P(t)/Q(t)$, where P and Q are algebraic polynomials, $Q(t) \neq 0$ in A, and P is of degree $2n - 2$. At least $n - 1$ points from the set $(\tau_j)_1^n$ are distinct from zero. Then, it follows from (6) that P has at least $2n - 1$ zeros and consequently $P(t) \equiv 0$, which leads to a contradiction with the assumption that $\varphi(t)$ is non-zero. The proof is completed.

It is seen by the same argument that $\{\varphi_i\}$ is actually an Extended Tchebycheff system on A.

Note that

$$B'(\bar{x}; t) = B(\bar{x}; t) \sum_{k=1}^{n} v_k(1 - x_k^2)\, \varphi_k(t) \qquad \text{for} \quad t \in A.$$

Then, it follows from Lemma 1 that $B'(\bar{x}; t)$ has exactly one (simple) zero t_i in (x_i, x_{i+1}), $i = 1, ..., n-1$, and these are all zeros of $B'(\bar{x}; t)$ in A. Thus, introducing the notations

$$\sigma_k := v_{k+1} + \cdots + v_n, \qquad k = 0, ..., n-1, \qquad \sigma_n := 0,$$

we have

$$\operatorname{sign} B(\bar{x}; t_k) = (-1)^{\sigma_k}, \qquad k = 0, ..., n, \tag{7}$$

where $t_0 = a$, $t_n = b$.

Next we prove a theorem about the existence of a Blaschke product having a preassigned shape.

THEOREM 1. *Let* $-1 < a < b < 1$. *Given* $(v_k)_1^n$ *and the numbers* $(h_k)_0^n$, *satisfying the conditions* $h_k \neq 0$,

$$\operatorname{sign} h_k = (-1)^{\sigma_k}, \qquad k = 0, ..., n,$$

there exists a unique system of points $(x_k)_1^n$ *and a constant* $c > 0$ *such that* $a < x_1 < \cdots < x_n < b$ *and*

$$cB(\bar{x}; t_k) = h_k, \qquad k = 0, ..., n,$$

where $(t_k)_1^{n-1}$ *are the zeros of* $B'(\bar{x}; t)$,

$$a = t_0 < x_1 < t_1 < \cdots < x_n < t_n = b.$$

Proof. According to the remark after Lemma 1, the zeros $(t_j)_1^{n-1}$ of $B'(\bar{x}; t)$ are uniquely determined by the points $\bar{x} = (x_1, ..., x_n)$. Moreover, by the implicit function theorem, $t_j = t_j(x_1, ..., x_n)$ is a differentiable function in $-1 < x_1 < \cdots < x_n < 1$. We are seeking a solution $c, x_1, ..., x_n$ of the system

$$f_k(c, x_1, ..., x_n) := cB(\bar{x}; t_k(x_1, ..., x_n)) = h_k, \qquad k = 0, ..., n. \tag{8}$$

Consider the Jacobian

$$J = \frac{D(f_0, ..., f_n)}{D(c, x_1, ..., x_n)}$$

of (8). We first prove that $\det J \neq 0$ at each point $(c, x_1, ..., x_n, h_0, ..., h_n)$ satisfying (8) with $h_0 \cdots h_n \neq 0$. In order to do this, note that

$$\frac{\partial f_k}{\partial c} = B(\bar{x}; t_k),$$

$$\frac{\partial f_k}{\partial x_j} = c\,\frac{\partial B(\bar{x}; t_k)}{\partial x_j} + c \sum_{i=1}^{n-1} \frac{\partial B(\bar{x}; t_k)}{\partial t_i} \cdot \frac{\partial t_i}{\partial x_j}$$
$$= cB(\bar{x}; t_k)\, v_j(t_k^2 - 1)\, \varphi_j(t_k).$$

Therefore, remembering that $cB(\bar{x}; t_k) = h_k$, we get

$$J = \begin{bmatrix} h_0/c & v_1(t_0^2-1)\, h_0 \varphi_1(t_0) \cdots v_n(t_0^2-1)\, h_0 \varphi_n(t_0) \\ \vdots & \vdots \qquad\qquad\qquad \vdots \\ h_k/c & v_1(t_k^2-1)\, h_k \varphi_1(t_k) \cdots v_n(t_k^2-1)\, h_k \varphi_n(t_k) \\ \vdots & \vdots \qquad\qquad\qquad \vdots \\ h_n/c & v_1(t_n^2-1)\, h_n \varphi_1(t_n) \cdots v_n(t_n^2-1)\, h_n \varphi_n(t_n) \end{bmatrix}.$$

Evidently,

$$\det J = c^{-1} \sum_{k=0}^{n} (-1)^k h_k \det \Delta_k, \tag{9}$$

where we have denoted by Δ_k the matrix obtained from J by deleting the first column and $(k+1)$st row. Further,

$$\det \Delta_k = (-1)^n v_1 \cdots v_n \prod_{\substack{j=0 \\ j \neq k}}^{n} h_j(1 - t_j^2) \cdot \det \Phi_k, \tag{10}$$

where

$$\Phi_k := \{\varphi_j(t_i)\}_{i=0, i\neq k,}^{n}{}_{j=1}^{n}.$$

Since $\{t_i\} \subset A$, it follows from Lemma 1 that $\det \Phi_k \neq 0$. One must determine the sign of $\det \Phi_k$. To do this, observe that

$$\det \Phi(\bar{x}, \bar{\tau}) := \det\{\varphi_j(\tau_i)\}_{i=1,}^{n}{}_{j=1}^{n}$$

is a continuous function of the parameters $(x_1, ..., x_n)$ and $(\tau_1, ..., \tau_n)$ in the domain

$$a \leqslant \tau_1 < x_1 < \cdots < x_{k-1} < \tau_k < x_k < x_{k+1} < \tau_{k+1} < \cdots < x_n < \tau_n \leqslant b,$$

which we denote here by Ω. Moreover, $\det \Phi(\bar{x}, \bar{\tau}) \neq 0$ in Ω and

$$\det \Phi(\bar{x}, \bar{\tau}) = \det \Phi_k \tag{11}$$

for $(\tau_1, ..., \tau_n) = (t_0, ..., t_{k-1}, t_{k+1}, ..., t_n)$. On the other hand, choosing τ_j very close to x_j $(j = 1, ..., n)$, $\Phi(\bar{x}, \bar{\tau})$ becomes a matrix with a dominant main diagonal. Thus

$$\text{sign} \det \Phi(\bar{x}, \bar{\tau}) = \text{sign} \prod_{j=1}^{n} (\tau_j - x_j) = (-1)^k$$

for this special choice of $\bar{x}$ and $\bar{\tau}$. Then by continuity, it follows from (11) that

$$\text{sign} \det \Phi_k = (-1)^k, \qquad k = 0, ..., n. \tag{12}$$

Using (12) in (10) we get from (9) that

$$\det J = (-1)^n v_1 \cdots v_n c^{-1} \prod_{j=0}^{n} h_j(1 - t_j^2) \sum_{k=0}^{n} |\det \Phi_k| \, (1 - t_k^2)^{-1}. \tag{13}$$

Therefore $\det J \neq 0$ if $h_0 \cdots h_n \neq 0$. Now let us return to the system (8). Clearly, for each $c^0 > 0$ and $a < x_1^0 < \cdots < x_n^0 < b$, there exist unique $(h_j^0)_0^n$ satisfying (8). h_j^0 is just the jth local extremum of $c^0 B(\bar{x}^0; t)$ in A. We fix some arbitrary $(c^0, \bar{x}^0)$ and consider the system (8) with a right-hand side $h_k(s) := s h_k + (1 - s) h_k^0$, i.e.,

$$c B(\bar{x}; t_k) = h_k(s), \qquad k = 0, ..., n. \tag{14}$$

Here s is a parameter in $[0, 1]$. The system (14) has a solution $(c^0, \bar{x}^0)$ for $s = 0$. Denote by $J(s)$ the Jacobian of (14). Since $\text{sign}\, h_k^0 = (-1)^{\sigma_k} = \text{sign}\, h_k$, it is clear that $h_k(s) \neq 0$ for $0 \leqslant s \leqslant 1$. Therefore

$$\det J(s) \neq 0 \qquad \text{for} \quad 0 \leqslant s \leqslant 1$$

at each point $(c, x_1, ..., x_n, h_1(s), ..., h_n(s))$ satisfying (14). It follows from (14) that c is bounded by an absolute constant. Then by the implicit function theorem (as in [5]), there exists a unique system of continuous functions $(c(s), \bar{x}(s))$ such that $c(0) = c^0$, $\bar{x}(0) = \bar{x}^0$, and $(c(s), \bar{x}(s))$ satisfies (14) for each $s \in [0, 1]$. Hence $(c(1), \bar{x}(1))$ is a solution of our system (8).

To establish the uniqueness of the solution we consider the mapping $\psi: \mathscr{X} \to \mathscr{X}$,

$$\mathscr{X} := \{(c, x_1, ..., x_n): c > 0, a < x_1 < \cdots < x_n < b\},$$

which transforms the starting points $(c^0, \bar{x}^0)$ into $(c(1), \bar{x}(1))$. Note that:

(a) ψ is a continuous map of $\mathscr{X}$ into itself.

Indeed, the functions $c(s)$, $\bar{x}(s)$ satisfy the equation

$$\frac{\partial f_k}{\partial c} \cdot \frac{dc}{ds} + \sum_{i=1}^{n} \frac{\partial f_k}{\partial x_i} \cdot \frac{dx_i}{ds} = h_k'(s), \qquad k = 0, ..., n, \tag{15}$$

in [0, 1]. In addition, the Jacobian of (15) is actually J and therefore non-zero. Then by a classical result from the theory of differential equations, $c(s)$, $\bar{x}(s)$ depend continuously on the initial conditions $c(0)=c^0$, $\bar{x}(0)=\bar{x}^0$.

(b) $\psi(\mathscr{X})$ consists of isolated points.

This follows directly from the fact that $\det J(1) \neq 0$.

Now, by property (a), $\psi(\mathscr{X})$ is connected. Then the second property (b) implies that $\psi(\mathscr{X})$ consists of only one point. The proof is completed.

The polynomial analogue of Theorem 1 was proved by the first author in [3]. Next Barrar and Loeb [2] extended the result to Tchebycheff systems modifying and improving the differential equations approach of Fitzgerald and Schumaker [5]. Our proof is based on their modification.

COROLLARY 1. *The coefficient c in* (8) *is a strictly increasing function of* $|h_k|$, $k=0, ..., n$.

Proof. Suppose that c satisfies (8). Then by the implicit function theorem,

$$\frac{\partial c}{\partial h_k} = \frac{(-1)^k \det \Delta_k}{\det J}.$$

Using (10), (12), and (13) one easily verifies that

$$\operatorname{sign} \frac{\partial c}{\partial h_k} = (-1)^{\sigma_k}.$$

This proves the assertion, since $\operatorname{sign} h_k = (-1)^{\sigma_k}$, as specified in Theorem 1.

We can now proceed to our main result.

THEOREM 2. *Given $[a, b] \subset (-1, 1)$ and multiplicities $(\nu_k)_1^n$, there exists a unique system of points $\bar{\xi} = (\xi_i)_1^n$, $a < \xi_1 < \cdots < \xi_n < b$, such that*

$$\|B(\bar{\xi}; \cdot)\| = \inf\{\|B(\bar{x}; \cdot)\| : a \leqslant x_1 < \cdots < x_n \leqslant b\}. \tag{16}$$

Moreover, the extremal Blaschke product $B(\bar{\xi}; t)$ is characterized by the property that there exist points $a = t_0 < t_1 < \cdots < t_{n-1} < t_n = b$ at which

$$B(\bar{\xi}; t_k) = (-1)^{\sigma_k} \|B(\bar{\xi}; \cdot)\|, \qquad k=0, ..., n.$$

Proof. It is not difficult to see that

$$\left|\frac{x-(a+\varepsilon)}{1-x(a+\varepsilon)}\right| < \left|\frac{x-a}{1-xa}\right|$$

on $a+\varepsilon<x<b$ for $0<\varepsilon<b-a$. Then

$$\|B(\bar{x}_\varepsilon;\cdot)\|<\|B(\bar{x};\cdot)\|$$

for sufficiently small $\varepsilon>0$ if $\bar{x}=(a, x_2, ..., x_n)$ and $\bar{x}_\varepsilon=(a+\varepsilon, x_2, ..., x_n)$. Thus we need to prove (16) in the set of those $\bar{x}$ for which $a<x_1$, $x_n<b$.

According to Theorem 1, there exist unique $c^*>0$ and $(\xi_i)_1^n$, $a<\xi_1<\cdots<\xi_n<b$, such that the Blaschke product $B(\bar{\xi}; t)$ satisfies the equations

$$c^*B(\bar{\xi}; t_k)=(-1)^{\sigma_k}, \qquad k=0, ..., n,$$

where $t_0=a$, $t_n=b$, and $(t_k)_1^{n-1}$ are the zeros of $B'(\bar{\xi}; t)$ in $(-1, 1)/\{\xi_1, ..., \xi_n\}$. Clearly

$$\|B(\bar{\xi};\cdot)\|=h^* := 1 \mid c^*.$$

Let us assume that some $B(\bar{x}; t)$ with $a<x_1<\cdots<x_n<b$ satisfies the inequality $\|B(\bar{x};\cdot)\|\leqslant h^*$. Then

$$|h_k|\leqslant h^*, \qquad k=0, ..., n,$$

where the $(h_k)_0^n$ are the corresponding local extremums of $B(\bar{x}; t)$ in $[a, b]/\{x_1, ..., x_n\}$. If $|h_k|=h^*$ for all k, then, according to Theorem 1, $\bar{\xi}=\bar{x}$ and therefore $B(\bar{x}; t)\equiv B(\bar{\xi}; t)$. Suppose that $|h_k|<h^*$ for at least one k. Since, by Corollary 1, c is an increasing function of $|h_k|$, we get

$$1=c(|h_0|, ..., |h_n|)<c(h^*, ..., h^*)=1,$$

a contradiction, which shows that

$$\|B(\bar{\xi};\cdot)\|<\|B(\bar{x};\cdot)\|$$

for each $a\leqslant x_1<\cdots<x_n\leqslant b$. The theorem is proved.

A. Pinkus treats in [8, p. 268] the simple node case of Theorem 2 (i.e., when $\nu_1=\cdots=\nu_n=1$). The uniqueness part of his elegant proof relies essentially on the fact that the nodes are simple. Here, as in many other extremal problems in approximation theory, the study of the multiple node case needs a certain deeper observation. Our proof of Theorem 2 is based on the strict monotonicity of the leading coefficient c. This property was observed and exploited first in [3].

3. Comparison Theorems

In view of the relation between the Blaschke products and the optimal recovery of functions in H^∞ on the basis of Hermitian data, Theorem 2 asserts the existence and uniqueness of optimal nodes of any fixed type $(\nu_1, ..., \nu_n)$.

We will show here how the error $E(\nu_1, ..., \nu_n)$ of the optimal recovery depends on the type $(\nu_1, ..., \nu_n)$ of the information data. To be precise,

$$E(\nu_1, ..., \nu_n) := \|B(\bar{\xi}; \cdot)\|,$$

where $\bar{\xi}$ are the extremal points described in Theorem 2.

THEOREM 3. *Let $(\nu_k)_1^n$ be arbitrary natural numbers and $[a, b] \subset (-1, 1)$. Then*

$$E(\nu_1, ..., \nu_{k-1}, \nu_k, ..., \nu_n) < E(\nu_1, ..., \nu_{k-1} + \nu_k, ..., \nu_n)$$

for each $2 \leqslant k \leqslant n$.

Proof. Let us note first that $B(\bar{x}; t)$ is a continuous function of its zeros $x_1, ..., x_n$ in $a \leqslant x_1 \leqslant \cdots \leqslant x_n \leqslant b$. Further, it is not difficult to see that $|x_k - x_{k+1}| \to 0$ as $|h_k| \to 0$ if other $|h_i|$ held fix (h_i being the local extremum of $B(\bar{x}; t)$ in (x_i, x_{i+1})). Since, by Theorem 1, each $c(\bar{h})\, B(\bar{x}(\bar{h}); t)$ is uniquely determined by its $\bar{h} = (h_0, ..., h_n)$, we conclude from the above-mentioned facts that $c(\bar{h})\, B(\bar{x}(\bar{h}); t)$ tends uniformly in $[a, b]$ to $c_0 B_0(t)$ as $h_k \to 0$ (h_i, $i \neq k$, remaining fix), where c_0 and the Blaschke product B_0 are defined by a system like (8) of n equations with parameters $\nu_1, ..., \nu_{k-2}$, $\nu_{k-1} + \nu_k$, $\nu_{k+1}, ..., \nu_n$ and a right-hand side $h_0, ..., h_{k-1}$, $h_{k+1}, ..., h_n$. Thus we can define $c(h_0, ..., h_{k-1}, 0, h_{k+1}, ..., h_n)$ as c_0, by continuity. Then, according to Corollary 1,

$$c(\bar{h}) > c_0 \qquad \text{if} \quad h_k > 0. \tag{17}$$

Let $B(t)$ and $B(\bar{\xi}; t)$ be the extremal Blaschke products (as in Theorem 2) for the parameters $(\nu_1, ..., \nu_{k-1} + \nu_{k+1}, ..., \nu_n)$ and $(\nu_1, ..., \nu_{k-1}, \nu_k, ..., \nu_n)$, respectively. Put, for convenience,

$$h^* := \|B(\bar{\xi}; \cdot)\|, \qquad h_i^* := (-1)^{\sigma_i} h^*, \qquad i = 0, ..., n.$$

Then, by (17),

$$1 = c(h_0^*, ..., h_n^*) > c(h_0^*, ..., h_{k-1}^*, 0, h_{k+1}^*, ..., h_n^*) =: c_0^*.$$

Since $\|B(\bar{\xi}; \cdot)\| = \|c_0^* B\|$, we see that

$$\|B(\bar{\xi}; \cdot)\| < \|B\|,$$

which was to be shown. The proof is completed.

Using Theorem 3 we can state the main assertion of Theorem 2 in the more general form

$$\|B(\bar{\xi};\cdot)\| = \inf\{\|B(\bar{x};\cdot)\| : a \leqslant x_1 \leqslant \cdots \leqslant x_n \leqslant b\}. \tag{18}$$

Denote by E_N the error $E(1, ..., 1)$ of the optimal recovery in the case of N simple nodes. As an immediate consequence of Theorem 3, we get

COROLLARY 2. *Let* $N = \nu_1 + \cdots + \nu_n$. *Then*

$$E_N \leqslant E(\nu_1, ..., \nu_n).$$

The equality is attained if and only if $\nu_1 = \cdots = \nu_n = 1$.

Thus the evaluation at the optimal points $x_1^*, ..., x_N^*$ of type $(1, ..., 1)$ is the best information in the class of all Hermitian type N evaluations. In the next theorem we show even more.

THEOREM 4. *Let* $l_1, ..., l_N$ *be arbitrary continuous linear functionals defined in* H^∞. *Then*

$$E_N \leqslant R(l_1, ..., l_N).$$

Proof. It was shown in [4] (see also [8]) that there exists a Blaschke product B with N zeros in D such that

$$l_k(B) = 0, \qquad k = 1, ..., N.$$

Next, by Proposition 4.6 of [8], $\|B^*\| \leqslant \|B\|$, where $B^*(t)$ is the extremal Blaschke product as in Theorem 2 for $n = N$ and $\nu_1 = \cdots = \nu_n = 1$. Now, an application of (3) gives

$$R(l_1, ..., l_N) \geqslant \|B\| \geqslant \|B^*\| = E_N,$$

which completes the proof.

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